

GRADIENT ESTIMATES FOR SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV GROWTH AND HARDY POTENTIAL

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ABSTRACT. This note is a continuation of the work [17]. We study the following quasilinear elliptic equations

$$-\Delta_p u - \frac{\mu}{|x|^p} |u|^{p-2} u = Q(x) |u|^{\frac{Np}{N-p}-2} u, \quad x \in \mathbb{R}^N,$$

where $1 < p < N, 0 \leq \mu < ((N-p)/p)^p$ and $Q \in L^\infty(\mathbb{R}^N)$. Optimal asymptotic estimates on the gradient of solutions are obtained both at the origin and at the infinity.

Keywords: Quasilinear elliptic equations; Hardy's inequality; Gradient estimate

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1. INTRODUCTION AND MAIN RESULT

Let $1 < p < N, 0 \leq \mu < \bar{\mu} = ((N-p)/p)^p$ and $p^* = Np/(N-p)$. In this note, we study the following quasilinear elliptic equations

$$-\Delta_p u - \frac{\mu}{|x|^p} |u|^{p-2} u = Q(x) |u|^{p^*-2} u, \quad x \in \mathbb{R}^N, \quad (1.1)$$

where

$$\Delta_p u = \sum_{i=1}^N \partial_{x_i} (|\nabla u|^{p-2} \partial_{x_i} u), \quad \nabla u = (\partial_{x_1} u, \dots, \partial_{x_N} u),$$

is the p -Laplacian operator and $Q \in L^\infty(\mathbb{R}^N)$.

Let $C_0^\infty(\mathbb{R}^N)$ be the space of smooth functions in \mathbb{R}^N with compact support and $\mathcal{D}^{1,p}(\mathbb{R}^N)$ the closure of $C_0^\infty(\mathbb{R}^N)$ in the seminorm $\|v\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = \|\nabla v\|_{L^p(\mathbb{R}^N)}$. A function $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a weak solution to equation (1.1) if

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \frac{\mu}{|x|^p} |u|^{p-2} u \varphi \right) = \int_{\mathbb{R}^N} Q(x) |u|^{p^*-2} u \varphi \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

In [17], the author obtained the following result on the asymptotic behaviors of solutions to equation (1.1) both at the origin and at the infinity.

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Theorem 1.1. *Let $Q \in L^\infty(\mathbb{R}^N)$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1). Then there exists a positive constant C depending on $N, p, \mu, \|Q\|_\infty$ and the solution u such that*

$$|u(x)| \leq C|x|^{-\gamma_1} \quad \text{for } |x| < R_0, \quad (1.2)$$

and that

$$|u(x)| \leq C|x|^{-\gamma_2} \quad \text{for } |x| > R_1,$$

where $0 < R_0 < 1 < R_1$ are constants depending on $N, p, \mu, \|Q\|_\infty$ and the solution u .

In the above theorem and in the following, the exponents γ_1 and γ_2 are defined as follows: consider the equation

$$(p-1)\gamma^p - (N-p)\gamma^{p-1} + \mu = 0, \quad \gamma \geq 0.$$

Due to our assumptions on N, p and μ , that is, $1 < p < N$ and $0 \leq \mu < \bar{\mu}$, above equation has two nonnegative solutions γ_1 and γ_2 and they satisfy

$$0 \leq \gamma_1 < \frac{N-p}{p} < \gamma_2 \leq \frac{N-p}{p-1}.$$

Note that the constants C, R_0, R_1 depend on the solution u . This dependence has been discussed in [17] in full details. Later in this note we will give a brief discussion on this dependence after giving our main result.

Asymptotic estimates for solutions to equation (1.1) and to its variants are useful. For applications of such estimates, we refer to e.g. [2, 4, 5, 7, 8, 12]. In the present note, we continue the work of [17] and study asymptotic behaviors of gradient of weak solutions to equation (1.1). Not much is known in this aspect.

To the best of our knowledge, all known results on the asymptotic behaviors of gradient of weak solutions to equation (1.1) are concerned with the special case in which $Q \equiv 1$. Let us discuss the known results according to the value of the parameter μ .

In the case when $\mu = 0$, a prototype of equation (1.1) when $Q \equiv 1$ is

$$-\Delta_p u = |u|^{p^*-2}u, \quad \text{in } \mathbb{R}^N. \quad (1.3)$$

When $p = 2$, Gidas, Ni and Nirenberg [13] proved that positive C^2 solutions of equation (1.3) (not necessarily in $\mathcal{D}^{1,2}(\mathbb{R}^N)$) satisfying

$$\liminf_{|x| \rightarrow \infty} (|x|^{N-2}u(x)) < \infty \quad (1.4)$$

must be of the form $u(x) = u_0^{\lambda, x_0}(x) = \lambda^{\frac{N-2}{2}}u_0(\lambda(x-x_0))$ for some $\lambda > 0$ and some $x_0 \in \mathbb{R}^N$, where

$$u_0(x) = (N(N-2))^{\frac{N-2}{4}} (1+|x|^2)^{-\frac{N-2}{2}}.$$

Hypothesis (1.4) was removed by Caffarelli, Gidas and Spruck in [3]. Thus for positive C^2 solutions u of equation (1.3) when $p = 2$, there exists $\lambda > 0$ and $x_0 \in \mathbb{R}^N$ such that $u = u_0^{\lambda, x_0}$. Hence we have that

$$\lim_{|x| \rightarrow 0} |\nabla u_0^{\lambda, x_0}(x)| |x| = 0,$$

and that

$$\lim_{|x| \rightarrow \infty} |\nabla u_0^{\lambda, x_0}(x)| |x|^{N-1} = C\lambda^{-\frac{N-2}{2}},$$

for some constant $C = C(N) > 0$.

In the general case when $p \in (1, N)$, we can follow the argument of [1, Theorem 3.13] to find that weak positive radial solutions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ to equation (1.3) are of the form $u(x) = u_0^{\lambda, x_0}(x) =$

$\lambda^{\frac{N-p}{p}} u_0(\lambda(x - x_0))$ for some $\lambda > 0$ and some $x_0 \in \mathbb{R}^N$, where $u_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a particular weak positive radial solution satisfying

$$\lim_{|x| \rightarrow 0} |\nabla u_0(x)| |x| = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} |\nabla u_0(x)| |x|^{\frac{N-1}{p-1}} = C$$

for some positive constant $C = C(N, p) > 0$. Thus for weak positive radial solution $u = u_0^{\lambda, x_0} \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, we have that

$$\lim_{|x| \rightarrow 0} |\nabla u_0^{\lambda, x_0}(x)| |x| = 0,$$

and that

$$\lim_{|x| \rightarrow \infty} |\nabla u_0^{\lambda, x_0}(x)| |x|^{\frac{N-1}{p-1}} = C \lambda^{-\frac{N-p}{p}},$$

for some positive constant $C = C(N, p) > 0$.

In the case when $\mu \in (0, \bar{\mu})$, a prototype of equation (1.1) when $Q \equiv 1$ is

$$-\Delta_p u - \frac{\mu}{|x|^p} |u|^{p-2} u = |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^N. \quad (1.5)$$

When $p = 2$, by Chou and Chu [10, Theorem B], every positive solution $u \in C^2(\mathbb{R}^N \setminus \{0\})$ must be radially symmetric with respect to the origin, provided that u satisfies

$$|x|^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} u(x) \in L_{loc}^\infty(\mathbb{R}^N). \quad (1.6)$$

Catrina and Wang [9] and Terracini [16] proved that every positive radial solution of equation (1.5) must be of the form $u(x) = u_0^\lambda(x) = \lambda^{\frac{N-2}{2}} u_0(\lambda x)$ for some $\lambda > 0$, where u_0 is given by

$$u_0(x) = (4N(\bar{\mu} - \mu)/(N - 2))^{\frac{N-2}{4}} \left(|x|^{\frac{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}}{\sqrt{\mu}}} + |x|^{\frac{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}}{\sqrt{\mu}}} \right)^{-\frac{N-2}{2}}.$$

Thus for positive solution u in $C^2(\mathbb{R}^N \setminus \{0\})$ satisfying (1.6), there is a constant $\lambda > 0$ such that $u(x) = u_0^\lambda(x) = \lambda^{\frac{N-2}{2}} u_0(\lambda x)$. From which, we have that

$$\lim_{|x| \rightarrow 0} |\nabla u_0^\lambda(x)| |x|^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu} + 1} = C_1 \lambda^{\sqrt{\bar{\mu} - \mu}},$$

and that

$$\lim_{|x| \rightarrow \infty} |\nabla u_0^\lambda(x)| |x|^{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu} + 1} = C_2 \lambda^{-\sqrt{\bar{\mu} - \mu}},$$

for some constants $C_1, C_2 > 0$ depending only on N and μ . We remark that, by (1.2) of Theorem 1.1, every weak solution $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ of equation (1.5) satisfies hypothesis (1.6).

In the general case when $p \in (1, N)$, Boumediene, Veronica and Peral [1, Theorem 3.13] proved that all weak positive radial solutions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ of equation (1.5) are of the form $u(x) = u_0^\lambda(x) = \lambda^{\frac{N-p}{p}} u_0(\lambda x)$ for some $\lambda > 0$, where u_0 is a particular weak positive radial solution in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ satisfying

$$\lim_{|x| \rightarrow 0} |\nabla u_0(x)| |x|^{\gamma_1 + 1} = C_1 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} |\nabla u_0(x)| |x|^{\gamma_2 + 1} = C_2, \quad (1.7)$$

for some constants $C_1, C_2 > 0$. Thus for any weak positive radial solution $u = u_0^\lambda$ of equation (1.5), we have that

$$\lim_{|x| \rightarrow 0} |\nabla u_0^\lambda(x)| |x|^{\gamma_1 + 1} = C_1 \lambda^{\frac{N-p}{p} - \gamma_1}, \quad (1.8)$$

and that

$$\lim_{|x| \rightarrow \infty} |\nabla u_0^\lambda(x)| |x|^{\gamma_2 + 1} = C_2 \lambda^{\frac{N-p}{p} - \gamma_2}, \quad (1.9)$$

with the constants $C_1, C_2 > 0$ given by (1.7).

In this note, we give the asymptotic estimates for the gradient of weak solutions to equation (1.1) both at the origin and at the infinity.

Theorem 1.2. *Let $Q \in L^\infty(\mathbb{R}^N)$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution of equation (1.1). Then there exists a positive constant C depending on $N, p, \mu, \|Q\|_\infty$ and u , such that*

$$|\nabla u(x)| \leq C|x|^{-\gamma_1-1} \quad \text{for } |x| < R_0, \quad (1.10)$$

and that

$$|\nabla u(x)| \leq C|x|^{-\gamma_2-1} \quad \text{for } |x| > R_1, \quad (1.11)$$

where $0 < R_0 < 1 < R_1$ depend on $N, p, \mu, \|Q\|_\infty$ and u .

Again, in the above theorem the positive constants C, R_0, R_1 depend on the solution u . Indeed, this is the case, since equation (1.1) when $Q \equiv 1$ is invariant under the scaling $v(x) = \lambda^{\frac{N-p}{p}} u(\lambda x)$, $\lambda > 0$. In above theorems and in the following, if we say a constant depends on the solution u , it means that the constant depends on $\|u\|_{p^*, \mathbb{R}^N}$, the L^{p^*} -norm of u , and also on the modulus of continuity of the function $h(r) = \|u\|_{p^*, B_r(0)} + \|u\|_{p^*, \mathbb{R}^N \setminus B_{1/r}(0)}$ at $r = 0$. Precisely, we can choose a constant $\epsilon > 0$ depending on N, p, μ and $\|Q\|_\infty$. Since $h(r) \rightarrow 0$ as $r \rightarrow 0$, there exists $r_0 > 0$ such that

$$\|u\|_{p^*, B_{r_0}(0)} + \|u\|_{p^*, \mathbb{R}^N \setminus B_{1/r_0}(0)} < \epsilon.$$

Then the constants C, R_0, R_1 in Theorem 1.1 and Theorem 1.2 depend also on r_0 . The reader is referred to find more details on this dependence in [17].

Estimates (1.8) and (1.9) imply that the exponents $\gamma_1 + 1$ and $\gamma_2 + 1$ in the estimates (1.10) and (1.11) respectively are optimal.

The idea to prove Theorem 1.2 is as follows. Let u be a weak solution to equation (1.1) and set

$$f(x) = \mu|x|^{-p}|u|^{p-2}u + Q(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N.$$

Then u is a weak solution to equation

$$-\Delta_p u = f \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (1.12)$$

For any ball $B_{|x|/2}(x)$ centered at x with radius $|x|/2$, $x \neq 0$, gradient estimate of the p -Laplacian equation (1.12) gives us

$$\sup_{B_{|x|/8}(x)} |\nabla u| \leq C \left(\int_{B_{|x|/4}(x)} |\nabla u|^p \right)^{\frac{1}{p}} + C|x|^{\frac{1}{p-1}} \|f\|_{\infty, B_{|x|/4}(x)}^{\frac{1}{p-1}}. \quad (1.13)$$

For the terms on the right hand side of (1.13), Theorem 1.1 gives estimates on the second term at the origin and at the infinity. The estimate on the first term follows from Caccioppoli inequality, see Lemma 2.2 in Section 2. So we obtain the estimates in Theorem 1.2 from (1.13).

The note is organized as follows. In Section 2, we prove Theorem 1.2. In Section 3 we prove the gradient estimate of p -Laplacian equation.

Our notations are standard. $B_R(x)$ is the open ball in \mathbb{R}^N centered at x with radius $R > 0$. We write

$$\int_{B_R(x)} u = \frac{1}{|B_R(x)|} \int_{B_R(x)} u,$$

where $|B_R(x)|$ is the n -dimensional Lebesgue measure of $B_R(x)$. Let Ω be an arbitrary domain in \mathbb{R}^N . We denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support in Ω . For any $1 \leq q \leq \infty$, $L^q(\Omega)$ is the Banach space of Lebesgue measurable functions u such that the norm

$$\|u\|_{q, \Omega} = \begin{cases} \left(\int_\Omega |u|^q \right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty \\ \text{esssup}_\Omega |f| & \text{if } q = \infty \end{cases}$$

is finite. The local space $L_{\text{loc}}^q(\Omega)$ consists of functions belonging to $L^q(\Omega')$ for all $\Omega' \subset\subset \Omega$. A function u belongs to the Sobolev space $W^{1,q}(\Omega)$ if $u \in L^q(\Omega)$ and its first order weak partial derivatives also belong to $L^q(\Omega)$. We endow $W^{1,q}(\Omega)$ with the norm

$$\|u\|_{1,q,\Omega} = \|u\|_{q,\Omega} + \|\nabla u\|_{q,\Omega}.$$

The local space $W_{\text{loc}}^{1,q}(\Omega)$ consists of functions belonging to $W^{1,q}(\Omega')$ for all open $\Omega' \subset\subset \Omega$. We recall that $W_0^{1,q}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{1,q,\Omega}$. For the properties of the Sobolev functions, we refer to the monograph [18].

2. PROOF OF MAIN RESULT

This section is devoted to the proof of Theorem 1.2. We need the following results. The first result is the gradient estimate for the p -Laplacian equation.

Proposition 2.1. *Let Ω be a domain in \mathbb{R}^N and $f \in L_{\text{loc}}^\infty(\Omega)$. Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ be a weak solution to equation*

$$-\Delta_p u = f \tag{2.1}$$

in Ω , that is,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Then for any ball $B_{2R}(x_0) \subset \Omega$, there holds

$$\sup_{B_{R/2}(x_0)} |\nabla u| \leq C \left(\int_{B_R(x_0)} |\nabla u|^p \right)^{\frac{1}{p}} + CR^{\frac{1}{p-1}} \|f\|_{\infty, B_R(x_0)}^{\frac{1}{p-1}}, \tag{2.2}$$

where $C > 0$ depends only on N and p .

Proposition 2.1 is well known. In the case $f \equiv 0$, Proposition 2.1 has been proved by DiBenedetto [11, Proposition 3]. We will follow the argument of DiBenedetto [11] to prove Proposition 2.1 in the next section.

The second result is a consequence of Theorem 1.1.

Lemma 2.2. *Let $Q \in L^\infty(\mathbb{R}^N)$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1). Let R_0, R_1 be the constants as in Theorem 1.1. Then there exists a positive constant C depending on $N, p, \mu, \|Q\|_\infty$ and the solution u such that*

$$\int_{B_{|x|/4}(x)} |\nabla u|^p \leq C |x|^{-p(\gamma_1+1)} \quad \text{for } 0 < |x| < R_0/2, \tag{2.3}$$

and that

$$\int_{B_{|x|/4}(x)} |\nabla u|^p \leq C |x|^{-p(\gamma_2+1)} \quad \text{for } |x| > 2R_1. \tag{2.4}$$

Proof. Fix $x \in \mathbb{R}^N$ such that $0 < |x| < R_0/2$. Let $B = B_{|x|/4}(x)$ and $2B = B_{|x|/2}(x)$. Let $\eta \in C_0^\infty(2B)$ be a cut-off function such that $0 \leq \eta \leq 1$ in $2B$ and $\eta \equiv 1$ on B , $|\nabla \eta| \leq 8/|x|$. Substituting test function $\varphi = \eta^p u$ into equation (1.1), we obtain that

$$\int_{2B} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{2B} \left(\frac{\mu}{|y|^p} \eta^p |u|^p + Q(y) \eta^p |u|^{p^*} \right).$$

We have that

$$\int_{2B} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \geq \frac{1}{2} \int_B |\nabla u|^p - C_p \int_{2B} |u|^p |\nabla \eta|^p$$

for some constant $C_p > 0$ depending only on p . Thus

$$\int_B |\nabla u|^p \leq C_p \int_{2B} \left(|u|^p |\nabla \eta|^p + \frac{\mu}{|y|^p} \eta^p |u|^p + Q(y) \eta^p |u|^{p^*} \right).$$

Applying (1.2) of Theorem 1.1, we obtain that

$$\int_B |\nabla u|^p \leq C |x|^{-p-p\gamma_1+N}, \quad \forall 0 < |x| < R_0/2,$$

where $C > 0$ depends on $N, p, \mu, \|Q\|_\infty$ and the solution u . This proves (2.3). We can prove (2.4) similarly. We finish the proof of Lemma 2.2. \square

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a solution to equation (1.1) with $Q \in L^\infty(\mathbb{R}^N)$. We only prove (1.10). We can prove (1.11) similarly. Let $R_0 \in (0, 1)$ be the constant as in Theorem 1.1. Set

$$f(x) \equiv \frac{\mu}{|x|^p} |u|^{p-2} u + Q(x) |u|^{p^*-2} u, \quad x \in B_{R_0}(0) \setminus \{0\}.$$

By (1.2) of Theorem 1.1 and the fact that $\gamma_1 < (N-p)/p$, we obtain that

$$|f(x)| \leq C |x|^{-p-(p-1)\gamma_1} \quad \forall 0 < |x| < R_0. \quad (2.5)$$

Thus $f \in L_{\text{loc}}^\infty(B_{R_0}(0) \setminus \{0\})$.

Since u is a weak solution to equation (1.1), u is a weak solution to equation (2.1) in $B_{R_0}(0) \setminus \{0\}$ with f given above. For any $x \in B_{R_0}(0) \setminus \{0\}$, we apply Proposition 2.1 on the ball $B_{|x|/2}(x)$ to obtain that

$$\sup_{B_{|x|/8}(x)} |\nabla u| \leq C \left(\int_{B_{|x|/4}(x)} |\nabla u|^p \right)^{\frac{1}{p}} + C |x|^{\frac{1}{p-1}} \|f\|_{\infty, B_{|x|/4}(x)}. \quad (2.6)$$

Combining (2.3), (2.5) and (2.6) gives that

$$\sup_{B_{|x|/8}(x)} |\nabla u| \leq C |x|^{-1-\gamma_1} \quad \forall 0 < |x| < R_0/2,$$

for some constant $C > 0$ depending on $N, p, \mu, \|Q\|_\infty$ and the solution u . This proves (1.10). \square

3. GRADIENT ESTIMATES FOR p -LAPLACIAN EQUATIONS

This section is devoted to the proof of Proposition 2.1.

Let $B_{2R}(x_0) \subset \Omega$ be an arbitrary ball. In the following we write $B_r = B_r(x_0)$ for all $r > 0$. Let $\epsilon > 0$. Following [11], we consider the equation

$$\begin{cases} -\operatorname{div} \left((\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} \nabla u_\epsilon \right) = f & \text{in } B_{2R}, \\ u_\epsilon = u & \text{on } \partial B_{2R}. \end{cases} \quad (3.1)$$

Then (3.1) admits a unique solution $u_\epsilon \in W^{1,p}(B_{2R})$ such that

$$u_\epsilon \in C^2(B_{2R}),$$

and up to a subsequence

$$u_\epsilon \rightarrow u \text{ and } \nabla u_\epsilon \rightarrow \nabla u \text{ uniformly in } B_R$$

as $\epsilon \rightarrow 0$.

To prove Proposition 2.1, we will prove the following estimate for u_ϵ :

$$\sup_{B_{R/2}} |\nabla u_\epsilon| \leq C \left(\int_{B_R} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} + CR^{\frac{1}{p-1}} \|f\|_{\infty, B_R}^{\frac{1}{p-1}}, \quad (3.2)$$

for a constant $C > 0$ depending only on N and p and independent of ϵ and R . Then by taking $\epsilon \rightarrow 0$ in (3.2), we obtain (2.2) and then Proposition 2.1 is proved.

We divide the proof of (3.2) into several lemmas. For simplicity, we write $v = u_\epsilon$ and $w = \epsilon + |\nabla v|^2$. We shall always assume that $N \geq 3$. We can prove (3.2) similarly when $N = 2$. First we derive the following Caccioppoli type inequality.

Lemma 3.1. *For any $\alpha \geq \max(p-2, 0)$ and any $\eta \in C_0^\infty(B_R)$, we have*

$$\int_{B_R} w^{\frac{\alpha+p-4}{2}} |\nabla w|^2 \eta^2 \leq C \int_{B_R} w^{\frac{\alpha+p}{2}} |\nabla \eta|^2 + C \int_{B_R} |f|^2 w^{\frac{\alpha+2-p}{2}} \eta^2, \quad (3.3)$$

for some $C = C(N, p) > 0$.

Proof. For simplicity, we write $\partial_i = \partial_{x_i}$, $\partial_{ij} = \partial_{x_i x_j}$ ($i, j = 1, \dots, N$). Differentiating equation (3.1) with respect to x_k ($k = 1, \dots, N$) gives

$$-\partial_i (\mathcal{A}^{ij} (\nabla v) \partial_{jk} v) = \partial_k f \quad \text{in } B_{2R},$$

where

$$\mathcal{A}^{ij} (\nabla v) = \left(\epsilon + |\nabla v|^2 \right)^{\frac{p-2}{2}} \delta_{ij} + (p-2) \left(\epsilon + |\nabla v|^2 \right)^{\frac{p-4}{2}} \partial_i v \partial_j v, \quad i, j = 1, \dots, N,$$

$\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The above equation is understood in the sense that, for all $\varphi \in C_0^\infty(B_{2R})$, (the summation notation is used throughout)

$$\int_{B_{2R}} \mathcal{A}^{ij} (\nabla v) \partial_{jk} v \partial_i \varphi = - \int_{B_{2R}} f \partial_k \varphi. \quad (3.4)$$

It is easy to prove that (3.4) holds also for all $\varphi \in W_0^{1,q}(B_{2R})$ for any $q \geq 1$. Set

$$\varphi = w^{\frac{\alpha}{2}} \partial_k v \eta^2,$$

where $\eta \in C_0^\infty(B_R)$ and $\alpha \geq \max(p-2, 0)$. Then

$$\partial_i \varphi = w^{\frac{\alpha}{2}} \partial_{ik} v \eta^2 + \frac{\alpha}{2} w^{\frac{\alpha}{2}-1} \partial_i w \partial_k v \eta^2 + 2w^{\frac{\alpha}{2}} \partial_k v \eta \partial_i \eta, \quad i = 1, \dots, N.$$

Substituting φ into equation (3.4), and summing up all $k = 1, \dots, N$, we obtain that

$$\begin{aligned} \sum_{k=1}^N \int_{B_R} \mathcal{A}^{ij} (\nabla v) \partial_{jk} v \partial_i \varphi &\geq C_1 \int_{B_R} w^{\frac{\alpha+p-2}{2}} |\nabla^2 v|^2 \eta^2 + C_1 (\alpha + p) \int_{B_R} w^{\frac{\alpha+p-4}{2}} |\nabla w|^2 \eta^2, \\ &- C_2 \int_{B_R} w^{\frac{\alpha+p}{2}} |\nabla \eta|^2, \end{aligned} \quad (3.5)$$

where $C_1, C_2 > 0$ depend only on p , $|\nabla^2 v| = \left(\sum_{i,j=1}^N (\partial_{ij} v)^2 \right)^{1/2}$, and that

$$\sum_{k=1}^N \left| \int_{B_R} f \partial_k \varphi \right| \leq \int_{B_R} |f| \left(w^{\frac{\alpha}{2}} |\nabla^2 v| \eta^2 + \alpha w^{\frac{\alpha-1}{2}} |\nabla w| \eta^2 + 2w^{\frac{\alpha+1}{2}} \eta |\nabla \eta| \right).$$

Applying Young's inequality

$$a^\theta b^{1-\theta} \leq \theta \delta a + \frac{1-\theta}{\delta^{\frac{\theta}{1-\theta}}} b, \quad \forall \delta, a, b > 0, \forall \theta \in [0, 1) \quad (3.6)$$

we obtain that

$$\begin{aligned} \sum_{k=1}^N \left| \int_{B_R} f \partial_k \varphi \right| &\leq \frac{C_1}{2} \int_{B_R} w^{\frac{\alpha+p-2}{2}} |\nabla^2 v|^2 \eta^2 + \frac{C_1(\alpha+p)}{2} \int_{B_R} w^{\frac{\alpha+p-4}{2}} |\nabla w|^2 \eta^2 \\ &\quad + C \int_{B_R} w^{\frac{\alpha+p}{2}} |\nabla \eta|^2 + C(\alpha+p) \int_{B_R} |f|^2 w^{\frac{\alpha+2-p}{2}} \eta^2. \end{aligned} \quad (3.7)$$

Combining (3.5) and (3.7), we obtain (3.3) for some $C > 0$ depends only on N and p . This finishes the proof of Lemma 3.1. \square

By the Sobolev inequality we obtain the following reverse inequality.

Lemma 3.2. *For any $\alpha \geq \max(p-2, 0)$ and $\eta \in C_0^\infty(B_R)$, we have*

$$\left(\int_{B_R} \left(\eta^2 w^{\frac{\alpha+p}{2}} \right)^\chi \right)^{1/\chi} \leq C_{N,p}(\alpha+p)^2 \left(\int_{B_R} w^{\frac{\alpha+p}{2}} |\nabla \eta|^2 + \int_{B_R} |f|^2 w^{\frac{\alpha+2-p}{2}} \eta^2 \right), \quad (3.8)$$

where $\chi = N/(N-2)$.

Proof. Let $h = \eta w^{\frac{\alpha+p}{4}}$. Then

$$|\nabla h|^2 \leq 2|\nabla \eta|^2 w^{\frac{\alpha+p}{2}} + (\alpha+p)^2 w^{\frac{\alpha+p-4}{2}} |\nabla w|^2 \eta^2.$$

By (3.3) of Lemma 3.1,

$$\int_{B_R} |\nabla h|^2 \leq C(\alpha+p)^2 \int_{B_R} \left(w^{\frac{\alpha+p}{2}} |\nabla \eta|^2 + |f|^2 w^{\frac{\alpha+2-p}{2}} \eta^2 \right), \quad (3.9)$$

where $C = C(N, p) > 0$. Now we use Sobolev inequality to obtain

$$\left(\int_{B_R} h^{2\chi} \right)^{\frac{1}{\chi}} \leq C_N \int_{B_R} |\nabla h|^2, \quad (3.10)$$

where $\chi = N/(N-2)$. Combining (3.9) and (3.10) yields (3.8). We finish the proof of Lemma 3.2. \square

In the following, we write

$$\bar{w} = w^{1/2} \quad \text{and} \quad F(r) = (r \|f\|_{\infty, B_r})^{1/(p-1)}. \quad (3.11)$$

As a consequence of Lemma 3.2, we have

Corollary 3.3. *Let $0 < r \leq R$ and $\alpha \geq \max(p-2, 0)$. Then for any $0 < r_1 < r_2 \leq r$, we have*

$$\left(\int_{B_{r_1}} \left(\bar{w}^{(\alpha+p)\chi} + F(r)^{(\alpha+p)\chi} \right) \right)^{\frac{1}{\chi}} \leq \frac{C_{N,p}(\alpha+p)^2}{(r_2 - r_1)^2} \int_{B_{r_2}} (\bar{w}^{\alpha+p} + F(r)^{\alpha+p}), \quad (3.12)$$

where $\chi = N/(N-2)$.

Proof. Let $\eta \in C_0^\infty(B_{r_2})$ be a cut-off function such that $0 \leq \eta \leq 1$ in B_{r_2} , $\eta \equiv 1$ on B_{r_1} and $|\nabla \eta| \leq 2/(r_2 - r_1)$. Substituting η into (3.8) we obtain that

$$\left(\int_{B_{r_1}} w^{\frac{(\alpha+p)\chi}{2}} \right)^{1/\chi} \leq C_{N,p}(\alpha+p)^2 \left(\frac{1}{(r_2 - r_1)^2} \int_{B_{r_2}} w^{\frac{\alpha+p}{2}} + \|f\|_{\infty, B_r}^2 \int_{B_{r_2}} w^{\frac{\alpha+2-p}{2}} \right).$$

Thus we have that

$$\left(\int_{B_{r_1}} w^{\frac{(\alpha+p)\chi}{2}} \right)^{1/\chi} \leq \frac{C_{N,p}(\alpha+p)^2}{(r_2 - r_1)^2} \left(\int_{B_{r_2}} w^{\frac{\alpha+p}{2}} + \int_{B_{r_2}} (r \|f\|_{\infty, B_r})^2 w^{\frac{\alpha+2-p}{2}} \right).$$

Since $\alpha \geq \max(p-2, 0)$ and $p > 1$, $\alpha + p > \alpha + 2 - p \geq 0$. Young's inequality (3.6) gives

$$(r\|f\|_{\infty, B_r})^2 w^{\frac{\alpha+2-p}{2}} \leq w^{\frac{\alpha+p}{2}} + (r\|f\|_{\infty, B_r})^{\frac{\alpha+p}{p-1}}.$$

Recall that \bar{w} and $F(r)$ are defined by (3.11). Thus we have

$$\begin{aligned} \left(\int_{B_{r_1}} w^{\frac{(\alpha+p)\chi}{2}} \right)^{1/\chi} &\leq \frac{C_{N,p}(\alpha+p)^2}{(r_2-r_1)^2} \left(\int_{B_{r_2}} w^{\frac{\alpha+p}{2}} + \int_{B_{r_2}} (r\|f\|_{\infty, B_r})^{\frac{\alpha+p}{p-1}} \right) \\ &= \frac{C_{N,p}(\alpha+p)^2}{(r_2-r_1)^2} \int_{B_{r_2}} (\bar{w}^{\alpha+p} + F(r)^{\alpha+p}). \end{aligned}$$

Since $r_1 < r_2 \leq r$, we have $|B_{r_1}|^{\frac{1}{\chi}} \leq C_N |B_{r_2}| / (r_2 - r_1)^2$. Therefore

$$\begin{aligned} \left(\int_{B_{r_1}} (\bar{w}^{(\alpha+p)\chi} + F(r)^{(\alpha+p)\chi}) \right)^{\frac{1}{\chi}} &\leq 2^{1-\frac{1}{\chi}} \left(\int_{B_{r_1}} (\bar{w}^{(\alpha+p)\chi}) \right)^{\frac{1}{\chi}} + 2^{1-\frac{1}{\chi}} F(r)^{\alpha+p} |B_{r_1}|^{\frac{1}{\chi}} \\ &\leq \frac{C_{N,p}(\alpha+p)^2}{(r_2-r_1)^2} \int_{B_{r_2}} (\bar{w}^{\alpha+p} + F(r)^{\alpha+p}) + \frac{C_N F(r)^{\alpha+p} |B_{r_2}|}{(r_2-r_1)^2}. \end{aligned}$$

Now it is easy to obtain that

$$\left(\int_{B_{r_1}} (\bar{w}^{(\alpha+p)\chi} + F(r)^{(\alpha+p)\chi}) \right)^{\frac{1}{\chi}} \leq \frac{C_{N,p}(\alpha+p)^2}{(r_2-r_1)^2} \left(\int_{B_{r_2}} (\bar{w}^{\alpha+p} + F(r)^{\alpha+p}) \right),$$

which gives (3.12). We finish the proof of Corollary 3.3. \square

Now we prove Proposition 2.1.

Proof of Proposition 2.1. We prove estimate (3.2). Let $\sigma \in (0, 1)$ and $0 < r \leq R$. Let $r_i = \sigma r + \frac{(1-\sigma)r}{2^i}$, $i = 0, 1, \dots$.

Case 1: $1 < p \leq 2$. In this case, define

$$\alpha_i = p\chi^i - p, \quad i = 0, 1, \dots.$$

Applying (3.12) with $r_1 = r_{i+1}$, $r_2 = r_i$ and $\alpha = \alpha_i$, we obtain that

$$M_{i+1} \leq \frac{C_{N,p}^{\frac{1}{p\chi^i}} (4\chi^2)^{\frac{i}{p\chi^i}}}{((1-\sigma)r)^{\frac{2}{p\chi^i}}} M_i, \quad i = 0, 1, \dots, \quad (3.13)$$

where

$$M_i = \left(\int_{B_{r_i}} (\bar{w}^{p\chi^i} + F(r)^{p\chi^i}) \right)^{\frac{1}{p\chi^i}},$$

and $\bar{w}, F(r)$ are defined by (3.11). An iteration of (3.13) gives us

$$M_{i+1} \leq \frac{C_{N,p}}{(1-\sigma)^{N/p} r^{N/p}} \left(\int_{B_r} (\bar{w}^p + F(r)^p) \right)^{\frac{1}{p}}.$$

Finally, letting $i \rightarrow \infty$, we obtain that

$$\sup_{B_{\sigma r}} (\bar{w} + F(r)) \leq \frac{C_{N,p}}{(1-\sigma)^{N/p}} \left(\int_{B_r} (\bar{w}^p + F(r)^p) \right)^{\frac{1}{p}} \leq \frac{C_{N,p}}{(1-\sigma)^{N/p}} \left(\left(\int_{B_r} \bar{w}^p \right)^{\frac{1}{p}} + F(r) \right).$$

In particular, choosing $\sigma = 1/2$ and $r = R$, we obtain (3.2) for $1 < p \leq 2$.

Case 2: $p > 2$. In this case, define

$$\alpha_i = (2p-2)\chi^i - p, \quad i = 0, 1, \dots.$$

Applying the same argument as above, we obtain that

$$\sup_{B_{\sigma r}} (\bar{w} + F(r)) \leq \frac{C_{N,p}}{(1-\sigma)^{N/(2p-2)}} \left(\int_{B_r} (\bar{w} + F(r))^{2p-2} \right)^{\frac{1}{2p-2}}.$$

Since $\sigma < 1$ and F is nondecreasing, we obtain that

$$\sup_{B_{\sigma r}} (\bar{w} + F(\sigma r)) \leq \frac{C_{N,p}}{(1-\sigma)^{N/(2p-2)}} \left(\int_{B_r} (\bar{w} + F(r))^{2p-2} \right)^{\frac{1}{2p-2}}. \quad (3.14)$$

Let $\sigma_i = 1 - \frac{1-\sigma}{2^i}$, $i = 0, 1, \dots$. Applying (3.14) with $r = \sigma_{i+1}R$, $\sigma = \sigma_i/\sigma_{i+1}$, we get that

$$M_i \leq \frac{C}{\left(1 - \frac{\sigma_i}{\sigma_{i+1}}\right)^{N\beta/p}} \left(\int_{B_R} (\bar{w} + F(R))^p \right)^{\frac{\beta}{p}} M_{i+1}^{1-\beta}, \quad (3.15)$$

where $\beta = p/(2p-2)$, and

$$M_i = \sup_{B_{\sigma_i R}} (\bar{w} + F(\sigma_i R)).$$

An iteration of (3.15) gives that

$$\sup_{B_{\sigma R}} (\bar{w} + F(\sigma R)) = M_0 \leq \frac{C_{N,p}}{(1-\sigma)^{N/p}} \left(\int_{B_R} (\bar{w} + F(R))^p \right)^{\frac{1}{p}}.$$

Choosing $\sigma = 1/2$, and applying Minkowski's inequality, we obtain (3.2) for $p > 2$. Thus we complete the proof of (3.2).

Now taking $\epsilon \rightarrow 0$ in (3.2), we obtain (2.2). Proposition 2.1 is proved. \square

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